Journal of Statistical Physics, Vol. 122, No. 2, January 2006 (© 2006)
DOI: 10.1007/s10955-005-8022-x

Improved Lower Bounds for the Critical Probability of Oriented Bond Percolation in Two Dimensions

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Received August 24, 2004; accepted July 27, 2005

We present a coupled decreasing sequence of random walks on Z that dominate the edge process of oriented bond percolation in two dimensions. Using the concept of *random walk in a strip*, we describe an algorithm that generates an increasing sequence of lower bounds that converges to the critical probability of oriented percolation p_c. From the 7th term on, these lower bounds improve upon 0.6298, the best rigorous lower bound at present, establishing 0.63328 as a rigorous lower bound for p_c. Finally, a *Monte Carlo* simulation technique is presented; the use thereof establishes 0.64450 as a non-rigorous five-digit-precision (lower) estimate for p_c.

KEY WORDS: Oriented percolation; Discrete time contact processes; Critical probability; Edge process; Markov chain in a strip; Coupling; Simulation.
Mathematics Subject Classification (1991): 60K35.

1. INTRODUCTION

Oriented percolation in two dimensions or discrete time contact process on $\mathbb{Z}$ is a family of discrete time stochastic processes in state space $\{0, 1\}^{\mathbb{Z}}$, indexed by a parameter $p \in [0, 1]$, the infection probability. It is well known that this family exhibits a phase transition, i.e. there exists a critical probability $p_c \in (0, 1)$ such that the processes die out with probability 1 if $p < p_c$, whereas the processes survive with positive probability if $p > p_c$. (See [1] or [2], for instance).

As usual in critical phenomena theory, an analytical expression for p_c is unknown, although numerical estimates abound in mathematical and physical

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[2] Supported by CNPq (grant N.301637/91-1).
[3] Supported by a grant from CNPq.

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literatures. At present, the best (rigorous) lower and upper bounds for p_c are 0.6298 and 2/3 respectively[(3)].

In this paper we present an algorithm that generates an increasing sequence of lower bounds that converges to the critical probability of oriented bond percolation in two dimensions. From the 7th term on, these lower bounds improve upon 0.6298, establishing 0.63328 as a rigorous lower bound for p_c.

More specifically:

1. a sequence of lower bounds for p_c, $\{p_c^{(i)}\}_{i\in\mathbb{N}}$, is generated through the following steps:
   (i) we construct in a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. we couple, a family of random walks $\{\,{}_p\overline{X}_n^{(i)}\}_{n\in\mathbb{N}}$ on $\mathbb{Z}$, indexed by $i \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $p \in (0, 1]$, together with a family of irreducible and aperiodic Markov chains $\{\,{}_pY_n^{(i)}\}_{n\in\mathbb{N}}$ taking values in $\mathcal{S}^{(i)} \overset{\text{def}}{=} \{0, 1\}^{\{2,4,\ldots,2i\}}$, indexed by $i \in \mathbb{N}$ and $p \in (0, 1)$, in such a way that
      (a) $(\forall\omega \in \Omega)\,(\forall i, j \in \overline{\mathbb{N}})\,(\forall p \in (0, 1])\,(\forall n \in \mathbb{N}) \qquad i \le j \Rightarrow {}_p\overline{X}_n^{(i)}[\omega] \ge {}_p\overline{X}_n^{(j)}[\omega]$
      (b) $(\forall\omega \in \Omega)\,(\forall i \in \overline{\mathbb{N}})\,(\forall p_1, p_2 \in (0, 1])\,(\forall n \in \mathbb{N}) \qquad p_1 \le p_2 \Rightarrow {}_{p_1}\overline{X}_n^{(i)}[\omega] \le {}_{p_2}\overline{X}_n^{(i)}[\omega]$
      (c) $\{\,{}_p\overline{X}_n^{(\infty)}\}_n$ corresponds to the right edge process of oriented percolation.[4]
      (d) the law of $(\,{}_p\overline{X}_{n+1}^{(i)} - {}_p\overline{X}_n^{(i)})$ on $\{\,{}_pY_n^{(i)} = \sigma\}$ does not depend on n (but it does depend on σ). This means that the Markov chain $\{\,{}_pY_n^{(i)}\}$ determines the *jump* distribution of the random walk $\{\,{}_p\overline{X}_n^{(i)}\}$.
      (e) ${}_p\overline{X}_n^{(i)}/n \xrightarrow{n} M^{(i)}(p) \overset{\text{def}}{=} \sum_{\sigma\in\mathcal{S}^{(i)}} \mathbb{E}^{\mathbb{P}}(\,{}_p\overline{X}_2^{(i)} - {}_p\overline{X}_1^{(i)} \mid {}_pY_1^{(i)} = \sigma) \cdot \pi(\sigma)$ *a.s.*, provided that π is the invariant measure of the Markov chain $\{\,{}_pY_n^{(i)}\}$. That is, the mean speed of the random walk $\{\,{}_p\overline{X}_n^{(i)}\}$ converges almost surely to its mean jump on configuration σ weighted according to the stationary measure of $\{\,{}_pY_n^{(i)}\}$.

   (ii) $p_c^{(i)}$ is defined to be the (only) value of p that nullifies $M^{(i)}(p)$, i.e. $M^{(i)}(p_c^{(i)}) = 0$. In words, when $p = p_c^{(i)}$, the random walk $\{\,{}_p\overline{X}_n^{(i)}\}$ has no drift in the long term.

[4] As defined in [2, pag.52], [1, sec.4] or in Section 2.3 of this paper, for example.

2. the sequence $\{p_c^{(i)}\}_{i\in\mathbb{N}}$ is shown to converge in non-decreasing fashion to p_c, i.e. $p_c^{(i)} \nearrow p_c$.
3. the first ten lower bounds $(p_c^{(0)}, p_c^{(1)}, ..., p_c^{(9)})$ are explicitly calculated, which improves on 0.6298 establishing 0.63328 as a rigorous lower bound for p_c.

The random walks alluded above and defined precisely in Section 2.3 correspond to the concept of *Random Walk in a (Half) Strip* developed in [4]. In strict sense, these processes are not proper random walks since their increments are neither stationary nor independent.

In the last section, we present a Monte Carlo simulation technique that exhibits a clear cut off between the subcritical and supercritical phases which enables a precise estimation of the critical probability of oriented percolation without the aid of scaling techniques. By means thereof, $p_c^{(1000)}$ was determined within a precision of 5 digits, that is $p_c \approx p_c^{(1000)} = (0.64451 \pm 0.00001)$.

2. DEFINITIONS AND CONSTRUCTIONS

Throughout this paper we adopt the following convention: $\mathbb{N} = \{0, 1, 2, \ldots\}$.

2.1. The Enviroment

$\mathcal{G} \equiv (\mathcal{V}, \mathcal{E})$ will denote the oriented graph having $\mathcal{V} = \{(n, m) \in \mathbb{N} \times \mathbb{Z} : (n + m)$ is even$\}$ as its set of vertices/sites and $\mathcal{E} = \{e^l_{nm}, e^r_{nm} : (n, m) \in \mathcal{V}\}$ as its set of bonds: bond e^l_{nm} points from site (n, m) to site $(n + 1, m - 1)$, whereas bond e^r_{nm} points from site (n, m) to site $(n + 1, m + 1)$. Sometimes the correspondence $l \leftrightarrow -1\ /\ r \leftrightarrow +1$ will be assumed throughout the text . We interpret n as a (discrete) time coordinate and m as a (discrete) space coordinate in $\mathcal{G}$ (see Fig. 1).

$\mathcal{V}_n$ denotes the nth slice of $\mathcal{V}$, i.e. $\mathcal{V}_n = \{(i, j) \in \mathcal{V} : i = n\}$, and $\mathbb{Z}_n$ the set of integers m such that $(m + n)$ is even.

It will be useful in the forthcoming definitions to identify $\mathcal{V}_n$ as $\mathbb{Z}_n$ and to think of $\{0, 1\}^{\mathcal{V}_n}$ as $\{0, 1\}^{\mathbb{Z}_n}$. Bearing in mind this identification and denoting a generic (spin) configuration in $\{0, 1\}^{\mathcal{V}_n}$ by η, we shall write $\eta(m)$ (instead of $\eta(n, m)$) to denote the spin of site (n, m) in configuration η.

2.2. The Probability Structure

Let $\{\xi^j_{nm} : (n, m) \in \mathcal{V}, j \in \{l, r\}\}$ be a family of independent and uniformly distributed (onto [0, 1]) random variables defined on the same abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Starting from this three-index family, we define the four-index family of *iid Bernoulli* random variables $\{{}_p\xi^j_{nm} :\ (n, m) \in \mathcal{V},\ \ j \in \{l, r\},\ \ p \in$

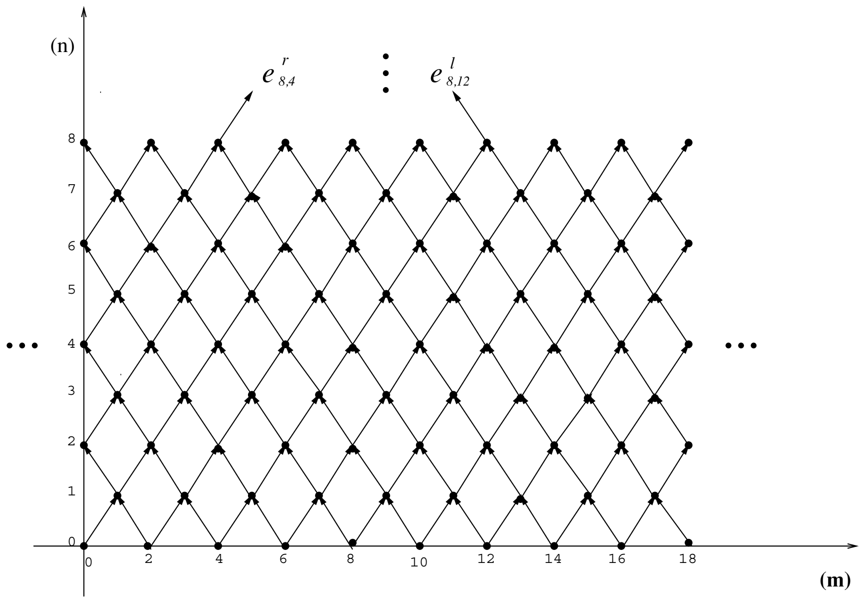

Fig. 1. Graph $\mathcal{G}$, whereon the processes will be defined (Sec. 2.3).

(0, 1]} by

$$ {}_{p}\xi^{j}_{nm} = \mathbf{1}_{\{\xi^{j}_{nm} \le p\}} \tag{1} $$

It follows straightforwardly from (1) that

$$ \begin{cases} \mathbb{P}\big({}_{p}\xi^{j}_{nm} = 1 \big) \ = \ p & (a) \\ \mathbb{P}\big({}_{p}\xi^{j}_{nm} = 0 \big) \ = \ 1 - p \ = \ q & (b) \end{cases} \tag{2} $$

and the *Fundamental Coupling Inequality:*

$$ (\forall \omega \in \Omega)\,(\forall (n,m) \in \mathcal{V})\,(\forall j \in \{l, r\}) \quad (p_1 \le p_2) \Rightarrow {}_{p_1}\xi^{j}_{nm}(\omega) \le {}_{p_2}\xi^{j}_{nm}(\omega) \tag{3} $$

${}_{p}\xi^{l}_{nm} = 1$ is interpreted as an open channel from site (n, m) to site $(n + 1, m - 1)$; ${}_{p}\xi^{r}_{nm} = 0$ as an obstructed channel from site (n, m) to site $(n + 1, m + 1)$; and so forth . . .

2.3. The Processes

2.3.1. The Strengthened Discrete Time Contact Processes (SDTCP)

In what follows, $\eta \in \{0, 1\}^{\mathcal{V}_n}$ will be interpreted as an infection state on slice $\mathcal{V}_n$ according to the rule:

$$\begin{cases} \eta(m) = 1\text{: site } m \text{ is infected at time } n \\ \eta(m) = 0\text{: site } m \text{ is healthy at time } n \end{cases} \tag{4}$$

Now, for each $i \in \overline{\mathbb{N}}$, $p \in (0, 1]$ and $\eta \in \{0, 1\}^{\mathcal{V}_0}$, we define the *Strengthened Discrete Time Contact Process* of i^{th}*-order*, *infection probability* p and *initial infection state*[5] η, $\{{}^{\eta}_{p}X^{(i)}_{n}\}_{n\in\mathbb{N}}$ by

Definition 2.1

(a) ${}^{\eta}_{p}X^{(i)}_{0} = \eta$, i.e. $(\forall m \in \mathbb{Z}_0)$ ${}^{\eta}_{p}X^{(i)}_{0}(m) = \eta(m)$

(b) ${}^{\eta}_{p}X^{(i)}_{n+1}(m) = \sup\{ {}^{\eta}_{p}X^{(i)}_{n}(m-1) \cdot {}_{p}\xi^{r}_{n,m-1};\ {}^{\eta}_{p}X^{(i)}_{n}(m+1) \cdot {}_{p}\xi^{l}_{n,m+1};$ $\mathbb{1}_{\{{}^{\eta}_{p}\overline{X}^{(i)}_{n+1} - m > 2i\}}\}$ $(\forall m \in \mathbb{Z}_{n+1})$
where ${}^{\eta}_{p}\overline{X}^{(i)}_{n+1} = \sup_{\substack{m\in\mathbb{Z}_n \\ j\in\{-1,+1\}}} \{(m+j) : {}^{\eta}_{p}X^{(i)}_{n}(m) \cdot \xi^{j}_{nm} = 1\}$

Remarks:

(i) The role of the indicator function in Definition 2.1(b) is to infect *by force* all the sites lying farther than $2i$ unit lengths on the left side of ${}^{\eta}_{p}\overline{X}^{(i)}_{n+1}$, the greatest infected site at time $n + 1$. Hence the name *Strengthened Discrete Time Contact Process* (SDTCP).

(ii) $(\forall n \in \mathbb{N})$ ${}^{\eta}_{p}X^{(i)}_{n}$ is a $\{0, 1\}^{\mathcal{V}_n}$-valued random variable.

(iii) The standard initial state to be assumed throughout this text is $\eta = \mathbf{1}_{\bullet\le 0} \in \{0, 1\}^{\mathcal{V}_0}$ defined by $\mathbf{1}_{\bullet\le 0}(m) = \mathbb{1}_{\{m\le 0\}}$. That is, unless otherwise stated, at time $n = 0$, all non-positive even sites will be assumed infected, whereas all positive even sites will be assumed healthy. In this case, we shall omit the initial condition. Therefore the symbols $\{{}_{p}X^{(i)}_{n}\}_{n\in\mathbb{N}}$, $\{{}^{\mathbf{1}_{\bullet\le 0}}_{p}X^{(i)}_{n}\}_{n\in\mathbb{N}}$, $\{{}_{p}X^{(i)}_{n}\}$, ${}_{p}X^{(i)}_{\bullet}$ share the same meaning and, for simplicity's sake, the last two ones will be prefered, provided that no confusion arises.

It follows directly from Definition 2.1 that

1. in the case of $i = \infty$, the indicator function never acts and we recover the ordinary *discrete time contact process*, also called *oriented percolation*

[5] Throughout this text we shall only consider initial configurations $\eta \in \{0, 1\}^{\mathcal{V}_0}$ such that $\eta(0) = 1$, $\eta(m) = 0, \forall m > 0$ and $\eta(m) = 1, \forall m < -2i$.

in two dimensions, as described in [1, Sec. 2]. Throughout this paper, we shall refer to it as $\{ {}_pX_n^{(\infty)}\}$;

2. the family $\{ {}^{\eta}_{p}X_n^{(i)}\}$ is decreasing in i in the following sense:

$$(\forall\omega\in\Omega)\,(\forall(n,m)\in\mathcal{V})\left(\forall i,j\in\overline{\mathbb{N}}\right)(\forall p\in(0,1])\left(\forall\eta\in\{0,1\}^{\mathcal{V}_0}\right)$$

$$i\le j\Rightarrow {}^{\eta}_{p}X_n^{(i)}(m)[\omega]\ge {}^{\eta}_{p}X_n^{(j)}(m)[\omega] \qquad (5)$$

In particular, the *oriented percolation process* ($i=\infty$) is the weakest of all.

3. the family $\{ {}^{\eta}_{p}X_n^{(i)}\}$ is increasing in p in the following sense:

$$(\forall\omega\in\Omega)\,(\forall(n,m)\in\mathcal{V})\left(\forall i\in\overline{\mathbb{N}}\right)(\forall r,s\in(0,1])\left(\forall\eta\in\{0,1\}^{\mathcal{V}_0}\right)$$

$$r\le s\Rightarrow {}^{\eta}_{r}X_n^{(i)}(m)[\omega]\le {}^{\eta}_{s}X_n^{(i)}(m)[\omega] \qquad (6)$$

4. the family $\{ {}^{\eta}_{p}X_n^{(i)}\}$ is increasing in η in the following sense:

$$(\forall\omega\in\Omega)\,(\forall(n,m)\in\mathcal{V})\left(\forall i\in\overline{\mathbb{N}}\right)(\forall p\in(0,1])\left(\forall\eta,\theta\in\{0,1\}^{\mathcal{V}_0}\right)$$

$$\eta\preceq\theta\Rightarrow {}^{\eta}_{p}X_n^{(i)}(m)[\omega]\le {}^{\theta}_{p}X_n^{(i)}(m)[\omega] \qquad (7)$$

where $\eta\preceq\theta$ means that $\eta(m)\le\theta(m)$ for all $m\in\mathbb{Z}_0$.

2.3.2. The Right Edge Processes (REP)

The second line of Definition 2.1(b) above defines the *right edge process*, a (non-markovian) random process on $\mathbb{Z}$ denoted by $\{ {}_p\overline{X}_n^{(i)}\}$ (or ${}_p\overline{X}_\bullet^{(i)}$ in abbreviated fashion). In words, ${}_p\overline{X}_n^{(i)}$ corresponds to the last infected site at time n. Accordingly, in Fig. 2 ${}_p\overline{X}_0^{(2)}=0$, ${}_p\overline{X}_1^{(2)}=-3$, ${}_p\overline{X}_2^{(2)}=-2$, ${}_p\overline{X}_3^{(2)}=-3$, ${}_p\overline{X}_4^{(2)}=-2$, ...

Again, in the case of $i=\infty$, the *edge process of oriented percolation* ${}_p\overline{X}_\bullet^{(\infty)}$ described in[1,2] is recovered.

It is useful to think of ${}_p\overline{X}_\bullet^{(i)}$ as random walks[6] on $\mathbb{Z}$.

2.3.3. The Induced Markov Chains

As a SDTCP ${}_pX_\bullet^{(i)}$ ($i<\infty$) evolves, an observer (sitting) at the right edge ${}_p\overline{X}_\bullet^{(i)}$ would notice a random evolution in the spins of the first i nearest neighboring sites to the left of him/her. This random process turns out to be a finite Markov

[6] Note, however, that the increments of ${}_p\overline{X}_\bullet^{(i)}$ are neither stationary nor independent.

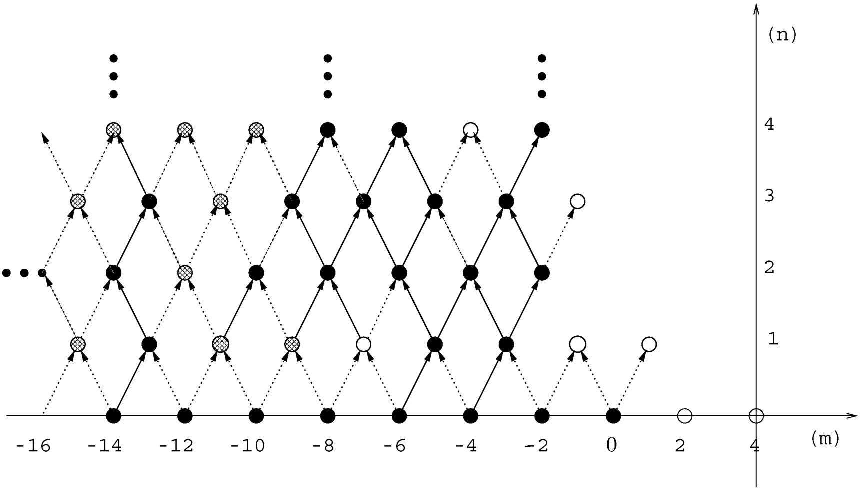

Fig. 2. A realization ω of the ${}_pX_\bullet^{(2)}$ process . Full-black arrows are open, whereas dotted arrows are closed to infection propagation. Black sites are infected, white sites are healthy and *gridded* sites were infected *by force*.

chain taking values in the space of the corresponding 2^i spin configurations. The next definition expresses this idea in mathematical terms:

Definition 2.2 The Markov chain $\{{}_pY_n^{(i)}\}_{n\in\mathbb{N}}$, $i \in \mathbb{N}$, with state space $\mathcal{S}^{(i)} = \{0,1\}^{\{2,4,\ldots,2i\}}$ defined by

$$ {}_pY_n^{(i)}(2j) = {}_pX_n^{(i)}\left({}_p\overline{X}_n^{(i)} - 2j\right), \quad 1 \le j \le i $$

is called *Induced Markov Chain* (IMC) associated to the SDTCP ${}_pX_\bullet^{(i)}$.
In what follows, a generic element $\sigma \in \mathcal{S}^{(i)}$ will be labeled by $l \in \{0, 1, \ldots, 2^i - 1\}$ according to the rule

$$ \sigma \leftrightarrow \sigma_l^{(i)} \Leftrightarrow l = \sum_{j=1}^{i} 2^{i-j}\sigma(2j) \tag{8} $$

For simplicity's sake, we shall omit the superscript (i), whenever no ambiguity arises. Accordingly, in Fig. 2 ${}_pY_0^{(2)} = \sigma_3$, ${}_pY_1^{(2)} = \sigma_2$, ${}_pY_2^{(2)} = \sigma_3$, ${}_pY_3^{(2)} = \sigma_3$, ${}_pY_4^{(2)} = \sigma_1, \ldots$

The transition probabilities of ${}_pY_\bullet^{(i)}$

$$ q_{lm}^{(i)}(p) \stackrel{\text{def}}{=} \mathbb{P}\left[\, {}_pY_{n+1}^{(i)} = \sigma_m \mid {}_pY_n^{(i)} = \sigma_l \right], \quad 0 \le l, m \le 2^i - 1 \tag{9} $$

turn out to be strictly positive polynomial functions of p, provided that $0 < p < 1$.[7] That is, ${}_pY_\bullet^{(i)}$ is an *irreducible* and *aperiodic* finite Markov chain. Hence $\pi^{(i)}(p)$, its stationary distribution on $\mathcal{S}^{(i)}$, is well defined.

The notation

$$\pi_l^{(i)}(p) \stackrel{\text{def}}{=} \pi^{(i)}(p)[\sigma_l], \qquad l = 0, 1, \ldots, 2^i - 1$$

is self-explanatory.

At this point, in accordance with [4, Sec.3.1], we state

Definition 2.3

$$P^{(i)}_{(l,m,k)}(p) \stackrel{\text{def}}{=} \mathbb{P}\big[\, {}_p\overline{X}^{(i)}_{n+1} = {}_p\overline{X}^{(i)}_n + (1-2k),\ {}_pY^{(i)}_{n+1} = \sigma_m \mid {}_pY^{(i)}_n = \sigma_l\big],$$
$$k \in \mathbb{N}, 0 \le l, m \le 2^i - 1$$

the transition probabilities from state σ_l to state σ_m with a jump of magnitude $(1-2k)$.

Remarks:

(i) each $P^{(i)}_{(l,m,k)}$ is a polynomial function of q.[8] In fact, it is readily seen by means of elementary combinatorics that each $P^{(i)}_{(l,m,k)}$ is a product of non-negative, integer powers of the *local transition probabilities* q^2, $(1-q^2)$, q, $(1-q)$, 1, 0 (see the table at the top of Fig. 3). More precisely, $P^{(i)}_{(l,m,k)}(q) = (q^2)^{n_1} \cdot (1-q^2)^{n_2} \cdot q^{n_3} \cdot (1-q)^{n_4} \cdot 1^{n_5} \cdot 0^{n_6}$, where each n_j, $1 \le j \le 6$ is a non-negative, integer function of (i, l, m, k) and 0^0 is interpreted as 1.

(ii) $P^{(i)}_{(l,m,k)}/P^{(i)}_{(l,m,k+1)} = 1/q^2$ for $k \ge i+2$. To see this, observe that a transition of type $(l, m, k+1)$ corresponds to a transition of type (l, m, k) shifted two unit lengths to the left. (Figure 3 ilustrates the case: $i = 2$, $l = 0$, $m = 2$, $k = 4$). Assuming ${}_p\overline{X}^{(i)}_n = 0$ for simplicity's sake, this means that the right edge must move backwards from $m = (1-2k)$ to $m = (1-2k-2)$ at time $n+1$. To accomplish this task, sites $(n, 2-2k)$ and $(n, -2k)$, which were infected/black[9] at time n, must not infect site $(n+1, 1-2k)$ (the former right edge's position when exposed to a transition of type (l, m, k)) through the edges $e^r_{n,-2k}$ and $e^l_{n,-2k+2}$ anymore. Since this event occurs with probability q^2 and all the remaining combinatorics remains unchanged, the desired result is established.

[7] See the remarks after Definition 2.3 for a detailed account thereof.
[8] The use of $q = 1 - p$ instead of p is due to algebraic simplicity.
[9] Remember that all sites at a distance larger than $2i$ to the left of the right edge are infected by force.

local transition	○ ● ●	● ● ●	○ ● ○	○ ○ ●	● ● ○	● ○ ●	○ ○ ○	● ○ ○
probability	q^2	$(1-q^2)$	q		$(1-q)$		1	0

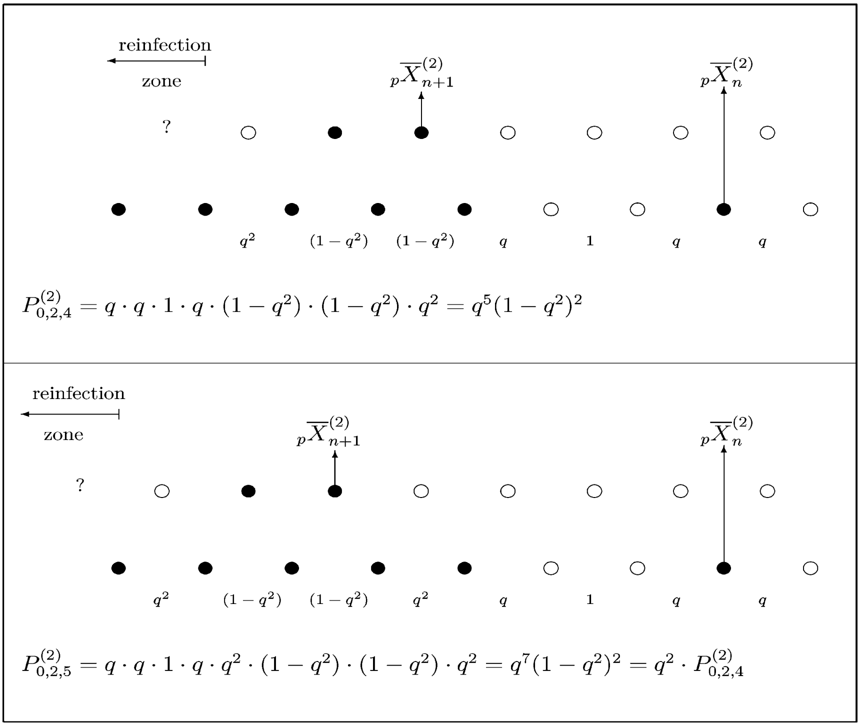

Fig. 3. Top: table of local transition probabilities. Upper half: a transition of type (0, 2, 4): $\sigma_0 \to \sigma_2$, $_p\overline{X}^{(2)}_{n+1} - {_p\overline{X}^{(2)}_{n}} = 1 - 2 \cdot 4 = -7$. Lower half: a transition of type (0, 2, 5): $\sigma_0 \to \sigma_2$, $_p\overline{X}^{(2)}_{n+1} - {_p\overline{X}^{(2)}_{n}} = 1 - 2 \cdot 5 = -9$.

(iii) Each transition probability $q^{(i)}_{lm}$, $0 \le l, m \le 2^i - 1$ may be expressed as $q^{(i)}_{lm} = \sum_{k=0}^{\infty} P^{(i)}_{(l,m,k)}$. Hence one may conclude from (i) and (ii) above that each $q^{(i)}_{lm}$ is a strictly positive polynomial[10] function of $q \in (0, 1)$.

[10] The polynomial character of $q^{(i)}_{lm}$ stems from the fact that each $P^{(i)}_{(l,m,k)}$ has at least one factor equal to $(1 - q^2)$, provided that $k \ge (i + 2)$.

Moreover, the 2^i entries of $(\pi_l^{(i)})$ turn out to be strictly positive rational functions of $q \in (0, 1)$.

Definition 2.4

$$M_l^{(i)}(p) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \sum_{m=0}^{2^i-1} (1-2k) \cdot P_{(l,m,k)}^{(i)}(p), \quad 0 \le l \le 2^i - 1$$

the mean jump/drift of the REP $\{ {}_p\overline{X}_n^{(i)} \}$ on configuration σ_l, i.e. $\mathbb{E}^{\mathbb{P}}({}_p\overline{X}_{n+1}^{(i)} - {}_p\overline{X}_n^{(i)} \mid {}_pY_n^{(i)} = \sigma_l)$
and

Definition 2.4

$$M^{(i)}(p) \stackrel{\text{def}}{=} \sum_{l=0}^{2^i-1} M_l^{(i)}(p) \cdot \pi_l^{(i)}(p) = \sum_{l=0}^{2^i-1} \mathbb{E}^{\mathbb{P}} \left({}_p\overline{X}_{n+1}^{(i)} - {}_p\overline{X}_n^{(i)} \middle| \, {}_pY_n^{(i)} = \sigma_l \right) \cdot \pi_l^{(i)}(p)$$

the asymptotic mean jump/drift of the REP $\{ {}_p\overline{X}_n^{(i)} \}$.

$M^{(i)}(p)$ will also be called the (asymptotic right) *edge speed* in accordance with Section 2.4 below. As it will soon become clear, Definition 2.5 is of fundamental importance in this paper.

Under this framework, the SDTCPs ${}_pX_\bullet^{(i)}, i < \infty$ can be regarded as *Markov Chains in a* 2^i*-row Strip*:

$${}_pX_n^{(i)} = \left({}_p\overline{X}_n^{(i)}, \ {}_pY_n^{(i)} \right) \tag{10}$$

a slightly different idea of *Markov Chains in a Half Strip* described in [4, Sec.3.1].

Remark: keeping remarks (i), (ii) and (iii) above in mind, it is not difficult to see that $M_l^{(i)}$, $0 \le l \le 2^i - 1$ and $M^{(i)}$ are rational functions of $q \in (0, 1)$ and hence of $p \in (0, 1)$.

2.4. Critical Probabilities

In Section 3 below it will be shown that, for $i \in \mathbb{N}$ and $p \in (0, 1]$:

(i) $M^{(i)}(p)$ is a scrictly incresing function of p;
(ii) $M^{(i)}(p)$ has only one real root in $(0, 1]$;
(iii) $\frac{{}_p\overline{X}_n^{(i)}}{n} \xrightarrow{(n)} M^{(i)}(p) \quad a.s.$

In the case of $i = \infty$, $\alpha(p)$, the (right) edge speed of oriented percolation, plays precisely the same role of $M^{(i)}(p)$ in the case of finite i depicted above[11]. (For details, see [1] for instance). Therefore the notation

$$M^{(\infty)}(p) \stackrel{\text{def}}{=} \alpha(p)$$

suggests itself and we state

Definition 2.6 The critical probability $p_c^{(i)}$ for the family of stochastic processes $\left\{ {}_pX_\bullet^{(i)} \right\}_{p\in(0,1]}$ is the only real root of the edge speed $M^{(i)}(p)$ in (0, 1]. Hence

$$M^{(i)}\left(p_c^{(i)}\right) = 0, \qquad \forall i \in \overline{\mathbb{N}}$$

Naturally $p_c^{(\infty)}$ corresponds to the critical probability of oriented percolation p_c alluded in Section 1

The heuristic meaning of Definition 2.6 is as follows:

(i) $p < p_c^{(i)} \Rightarrow \lim_{n\to\infty} {}_p\overline{X}_n^{(i)} = -\infty \quad a.s.$, i.e. the infection disappears with probability one;

(ii) $p > p_c^{(i)} \Rightarrow \lim_{n\to\infty} {}_p\overline{X}_n^{(i)} = +\infty \ a.s.$, i.e. the infection spreads out over all $\mathbb{Z}$.

In the sequel, we prove an important relation concerning the critical probabilities just defined, viz. $p_c^{(i)} \nearrow p_c^{(\infty)}$ as $i \to \infty$.

This non-decreasing convergence to the critical probability of oriented percolation (to be called *The Convergence Theorem*) besides the possibility of calculating the *finite* ($i \in \mathbb{N}$) critical probabilities by algebraical means (Section 4 below) are the cornerstone of this work.

3. THE CONVERGENCE THEOREM AND PRELIMINARY RESULTS

Lemma 3.1

$$(\forall i \in \overline{\mathbb{N}})(\forall p \in (0,1]) \ \lim_{n\to\infty} \frac{{}_p\overline{X}_n^{(i)}}{n} = M^{(i)}(p) \ \text{a.s.}$$

Proof:

First case ($i \in \mathbb{N}$): Let $n_l^{(i)}$ be the (random) number of visits that the IMC ${}_pY_\bullet^{(i)}$ makes to state σ_l up to time n (so that $\sum_{l=0}^{2^i-1} n_l^{(i)} = n$) and $J_{l,k}^{(i)}$ the kth

[11] $\alpha(p) = -\infty$, whenever $p < p_c$; thus the strictly increasing behaviour of $M^{(i)}(p)$ in the case of finite i does not apply to $\alpha(p)$ precisely.

jump of the REP ${}_p\overline{X}^{(i)}_\bullet$ on state/row σ_l. The (strong) Markov property of SDTCP ${}_pX^{(i)}_\bullet$ makes $J^{(i)}_{l,k}, : \ k \in \{1, 2, 3, \ldots\}$ *iid* random variables with $\mathbb{E}[J^{(i)}_{l,k}] = M^{(i)}_l$. Now,

$$\frac{{}_p\overline{X}^{(i)}_n}{n} = \sum_{l=0}^{2^i-1} \sum_{k=1}^{n_l^{(i)}} \frac{J^{(i)}_{l,k}}{n} = \sum_{l=0}^{2^i-1} \sum_{k=1}^{n_l^{(i)}} \frac{J^{(i)}_{l,k}}{n_l^{(i)}} \cdot \frac{n_l^{(i)}}{n} = \sum_{l=0}^{2^i-1} \frac{n_l^{(i)}}{n} \cdot \sum_{k=1}^{n_l^{(i)}} \frac{J^{(i)}_{l,k}}{n_l^{(i)}}, \quad (11)$$

$$\lim_{n\to\infty} \frac{n_l^{(i)}}{n} = \pi_l^{(i)} \ a.s. \quad \text{and} \quad \lim_{n\to\infty} \sum_{k=1}^{n_l^{(i)}} \frac{J^{(i)}_{l,k}}{n_l^{(i)}} = M_l^{(i)}(p) \ a.s.$$

Taking limits ($n \to \infty$) on both extremities of (11) and bearing in mind Definition 2.5, the desired result is established.

Second case ($i = \infty$): See [1, pag.1004]. □

Lemma 3.2 *For all finite i, the mean drift functions* $M^{(i)}(p) : (0, 1] \to (-\infty, 1]$ *are strictly increasing in p, i.e.* $(\forall i \in \mathbb{N})(\forall p_1, p_2 \in (0, 1])\ \ p_1 < p_2 \Rightarrow M^{(i)}(p_1) < M^{(i)}(p_2)$. *Moreover,* $M^{(i)}$ *is a surjection from* $(0, 1]$ *into* $(-\infty, 1]$.

Proof: The non-decreasing behaviour of $M^{(i)}(p)$ follows from inequality (6) above and lemma 3.1 just established. Definitions 2.2, 2.3, 2.4, 2.5 and the corresponding remarks make $M^{(i)}(p)$ a rational function of p such that $M^{(i)}(0_+) = -\infty$ and $M^{(i)}(1) = 1$; thus $M^{(i)}(p)$ can not be constant on any interval $[p_1, p_2] \subset (0, 1]$ and the strict behaviour follows.[12] □

Commentary on Lemma 3.2: It is worth observing that the function $\alpha(p) \stackrel{\text{def}}{=} M^{(\infty)}(p)$ is non-decreasing, strictly positive on $(p^c, 1)$, null at $p = p_c$ and infinitely negative on $(0, p_c)$. Again, the reference is.[(1)]

Lemma 3.2 above yields

Corollary 3.3 *For each* $i \in \overline{\mathbb{N}}$, $M^{(i)}$ *has only one real root in (0,1], denoted by* $p_c^{(i)}$, *in accordance with Section 2.4*

[12] To see why $M^{(i)}(0_+) = -\infty$, observe that ${}_p\overline{X}^{(i)}_1 \searrow -\infty\ a.s.$ as $p \searrow 0$; apply the Monotone Convergence Theorem to conclude that $\mathbb{E}({}_p\overline{X}^{(i)}_1) \searrow -\infty$ as $p \searrow 0$; keep in mind inequality (7) and the Markov property of ${}_pX^{(i)}_\bullet$ to yield $M_l^{(i)}(p) \stackrel{\text{def}}{=} \mathbb{E}({}_p\overline{X}^{(i)}_{n+1} - {}_p\overline{X}^{(i)}_n | Y_n^{(i)} = \sigma_l) \searrow -\infty$ as $p \searrow 0$ and finally establish that $M^{(i)}(p) \stackrel{\text{def}}{=} \sum_{l=0}^{2^i-1} M_l^{(i)}(p)\pi_l^{(i)}(p) \searrow -\infty$ as $p \searrow 0$.

Lemma 3.4 *The sequence* $\{M^{(i)}(p)\}_{i\in\overline{\mathbb{N}}}$ *is non-increasing, that is* $(\forall i, j \in \overline{\mathbb{N}})$ $(\forall p \in (0, 1])$ $i \le j \Rightarrow M^{(i)}(p) \ge M^{(j)}(p)$. *In particular* $(\forall i \in \mathbb{N})$ $(\forall p \in (0, 1])$ $\alpha(p) = M^{(\infty)}(p) \le M^{(i)}(p)$.

Proof: The non-increasing behaviour of $\left\{M^{(i)}(p)\right\}_{i\in\overline{\mathbb{N}}}$ follows from inequality (5) above and again from Lemma 3.1. □

Lemma 3.4 yields

Corollary 3.5 *The numerical sequence* $\{p_c^{(i)}\}_{i\in\mathbb{N}}$ *is non-decreasing. Moreover, for all finite* i, $p_c = p_c^{(\infty)} \ge p_c^{(i)}$. *Hence* $p_c \ge \lim_{i\to\infty} p_c^{(i)}$.

At this point, we turn our attention to the reverse (and more difficult) inequality $p_c \le \lim_{i\to\infty} p_c^{(i)}$. For that purpose, we suppose our abstract probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is large enough to support an additional family of random variables $\{\varsigma_{p,i} : p \in (0, 1],\ i \in \mathbb{N}\}$ independent of the family $\{\xi^j_{nm} : j \in \{l, r\}, (n, m) \in \mathcal{V}\}$ (subsection 2.2) and such that each $\varsigma_{p,i}$ is marginally distributed in $\mathcal{S}^{(i)} = \{\sigma_l,\ 0 \le l \le 2^i - 1\}$, the space of local spin configurations defined in Section 2.3.3, according to $\pi^{(i)}(p)$.[13]

Now, to each $\varsigma_{p,i}$ we associate the random variable $\widehat{\varsigma}_{p,i}$ taking values in $\{0, 1\}^{\mathcal{V}_0}$ and defined by

$$\widehat{\varsigma}_{p,i}(2m) \stackrel{\text{def}}{=} \begin{cases} 0, & m > 0 \\ \varsigma_{p,i}(-2m), & -i \le m \le -1 \\ 1, & m < -i \end{cases}$$

By analogy, we make the association $\mathcal{S}^{(i)} \ni \sigma \leftrightarrow \widehat{\sigma} \in \{0, 1\}^{\mathcal{V}_0}$.

Keeping in mind the foregoing paragraphs, for each finite i and $p \in (0, 1]$, we define the following stochastic processes on $(\Omega, \mathcal{F}, \mathbb{P})$ by analogy with subsection 2.3:

$$\text{(a)}\ {}^{\pi}_{p}X^{(i)}_{\bullet} \stackrel{\text{def}}{=} {}^{\widehat{\varsigma}_{p,i}}_{p}X^{(i)}_{\bullet} \qquad \text{(b)}\ {}^{\pi}_{p}Y^{(i)}_{\bullet} \stackrel{\text{def}}{=} {}^{\widehat{\varsigma}_{p,i}}_{p}Y^{(i)}_{\bullet} \qquad \text{(c)}\ {}^{\pi}_{p}\overline{X}^{(i)}_{\bullet} \stackrel{\text{def}}{=} {}^{\widehat{\varsigma}_{p,i}}_{p}\overline{X}^{(i)}_{\bullet} \tag{12}$$

To put it differently, (12) means that ${}^{\pi}_{p}X^{(i)}_{\bullet} = {}^{\widehat{\sigma}_l}_{p}X^{(i)}_{\bullet}$ on $\{\varsigma_{p,i} = \sigma_l\}$; that is, ${}^{\pi}_{p}X^{(i)}_{\bullet}$ chooses a random local spin configuration $\varsigma_{p,i}$ in $\mathcal{S}^{(i)}$ according to the law $\pi^{(i)}(p)$, adopts $\widehat{\varsigma}_{p,i} \in \{0, 1\}^{\mathcal{V}_0}$ as its initial configuration and then evolves according to the dynamics prescribed in Definition 2.1

Since ${}^{\pi}_{p}Y^{(i)}_{\bullet}$ starts from its stationary distribution $\pi^{(i)}(p)$, it is a stationary Markov chain, i.e. $(\forall n \in \mathbb{N})(\forall l \in \{0, 1, \dots, 2^i - 1\})\ \mathbb{P}(\{{}^{\pi}_{p}Y^{(i)}_{n} = \sigma_l\}) = \pi^{(i)}_l(p)$.

[13] In fact, $(\Omega, \mathcal{F}, \mathbb{P})$ need only support an additional independent uniform random variable ξ, since then we can define each $\varsigma_{p,i}$ as an appropriate simple deterministic function of ξ.

Hence we may conclude that ${}^{\pi}_{p}\overline{X}^{(i)}_{\bullet}$ has stationary increments such that $(\forall n \in \mathbb{N})\, \mathbb{E}({}^{\pi}_{p}\overline{X}^{(i)}_{n+1} - {}^{\pi}_{p}\overline{X}^{(i)}_{n}) = M^{(i)}(p)$.

At this point, we are able to prove

Lemma 3.6 $(\forall n, i \in \mathbb{N})\,(\forall p \in (0, 1])\ \mathbb{E}({}_{p}\overline{X}^{(i)}_{n}) \geq n \cdot M^{(i)}(p)$.

Proof: According to inequality (7) and (12) above

$$(\forall \omega \in \Omega)(\forall n, i \in \mathbb{N})(\forall p \in (0, 1])\ {}^{\pi}_{p}\overline{X}^{(i)}_{n}[\omega] \stackrel{\text{def}}{=} {}^{\widehat{\varsigma}_{p,i}}_{p}\overline{X}^{(i)}_{n}[\omega] \leq {}_{p}\overline{X}^{(i)}_{n}[\omega] \quad (13)$$

Integrating both sides of (13) and keeping in mind the stationarity of the increments of ${}^{\pi}_{p}\overline{X}^{(i)}_{\bullet}$ yields the desired result:

$$\mathbb{E}\left({}_{p}\overline{X}^{(i)}_{n}\right) \geq \mathbb{E}\big({}^{\pi}_{p}\overline{X}^{(i)}_{n}\big) = n \cdot M^{(i)}(p)$$

□

Lemma 3.7 $(\forall n \in \mathbb{N})(\forall p \in (0, 1])\ \lim_{i\to\infty} \mathbb{E}({}_{p}\overline{X}^{(i)}_{n}) = \mathbb{E}({}_{p}\overline{X}^{(\infty)}_{n})$

Proof: According to Definition 2.1, for all $i \in \overline{\mathbb{N}}$ and $p \in (0, 1]$, the corresponding right edge process ${}_{p}\overline{X}^{(i)}_{\bullet}$ can move at most one site to the right at each single time interval, that is $n \geq {}_{p}\overline{X}^{(0)}_{n} \geq {}_{p}\overline{X}^{(1)}_{n} \geq \ldots \geq {}_{p}\overline{X}^{(\infty)}_{n}$. Since infections *by force* with regard to the SDTCP ${}_{p}X^{(i)}_{\bullet}$ can only occur at a (even) distance (strictly) greater than $2i$ unit-lengths to the left of the right edge position ${}_{p}\overline{X}^{(i)}_{n}$, we can conclude that no infection *by force* can occur in the region $\mathcal{R}_i \stackrel{\text{def}}{=} \{(n, m) \in \mathcal{V} : m \geq n - 2i\}$ as regards ${}_{p}X^{(i)}_{\bullet}$. (See the case $i = 4$ in Fig. 4). Hence we can see (by means of finite induction on n) that ${}_{p}X^{(i)}_{\bullet}$ and ${}_{p}X^{(\infty)}_{\bullet}$ are indistinguishable over $\mathcal{R}_i$, that is

$$(\forall i \in \mathbb{N})\,(\forall p \in (0, 1])\,(\forall (n, m) \in \mathcal{V})\ (n, m) \in \mathcal{R}_i \Rightarrow {}_{p}X^{(i)}_{n}(m) = {}_{p}X^{(\infty)}_{n}(m) \quad (14)$$

An imediate consequence of (14) is

$$\begin{aligned}(\forall n \in \mathbb{N})\quad \left\{{}_{p}\overline{X}^{(\infty)}_{n} \neq {}_{p}\overline{X}^{(i)}_{n}\right\} &= \left\{{}_{p}\overline{X}^{(\infty)}_{n} < {}_{p}\overline{X}^{(i)}_{n}\right\} \subset \left\{{}_{p}\overline{X}^{(\infty)}_{n} < n - 2i\right\} \\ &= \bigcap_{\substack{m\in\mathbb{Z}_n,\\ m\geq n-2i}} \left\{{}_{p}X^{(\infty)}_{n}(m) = 0\right\} \qquad (15)\end{aligned}$$

where the last event in (15) amounts to saying that *oriented percolation* does not infect slice $\mathcal{V}_n$ to the right of site $(n, n - 2i)$, that is, all sites in $\mathcal{V}_n \cap \mathcal{R}_i$ must be healthy/white (an illustration thereof, where $n = 5$ and $i = 4$, can be seen in Fig. 4).

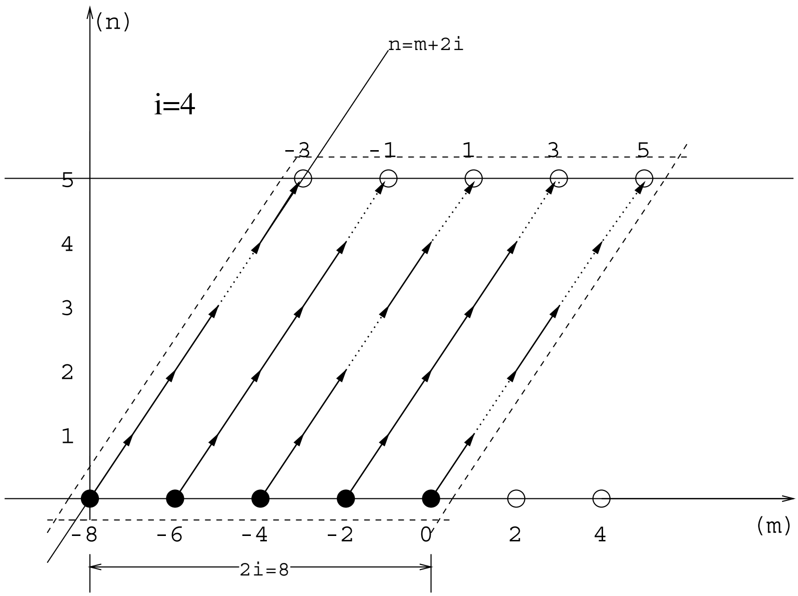

Fig. 4. No infection *by force* can occur in the region inside the dashed parallelogram. The processes ${}_pX_\bullet^{(4)}$ and ${}_pX_\bullet^{(\infty)}$ are indistinguishable therein.

Now, observe that, if *oriented percolation* is not present in the region $\mathcal{V}_n \cap \mathcal{R}_i$, all paths joining the sites therein to the non-positive side of slice $\mathcal{V}_0$, i.e. $\{(0, 2m) \in \mathcal{V}_0 : m \leq 0\}$, must be obstructed somewhere. In particular, all $(i+1)$ straight lines joining site $(0, -2j)$ to site $(n, n-2j)$, $0 \leq j \leq i$ must be interrupted at some point (Fig.4). Since these lines are made of different, independent bonds, the probability of this last event equals $(1-p^n)^{i+1}$. Thus

$$\mathbb{P}\left(\left\{ {}_p\overline{X}_n^{(i)} \neq {}_p\overline{X}_n^{(\infty)} \right\}\right) \leq \mathbb{P}\left[\bigcap_{\substack{m \in \mathbb{Z}_n,\\ m \geq n-2i}} \left\{ {}_pX_n^{(\infty)}(m) = 0 \right\} \right] \leq (1-p^n)^{i+1} \tag{16}$$

That is, ${}_p\overline{X}_n^{(i)} \overset{i}{\longrightarrow} {}_p\overline{X}_n^{(\infty)}$ in probability. Hence there is a subsequence $({}_p\overline{X}_n^{(i_k)})_{k \in \mathbb{N}}$ such that ${}_p\overline{X}_n^{(i_k)} \overset{k}{\longrightarrow} {}_p\overline{X}_n^{(\infty)}$ *almost surely* [5, Theorem 7.6]. Since the whole sequence $({}_p\overline{X}_n^{(i)})_{i \in \mathbb{N}}$ is non increasing thanks to inequality (5), we must have ${}_p\overline{X}_n^{(i)} \overset{i}{\searrow} {}_p\overline{X}_n^{(\infty)}$ *almost surely* as well, which amounts to the *almost sure* convergence $0 \leq (n - {}_p\overline{X}_n^{(i)}) \overset{i}{\nearrow} (n - {}_p\overline{X}_n^{(\infty)})$. On the other hand, thanks to the second inequality in (16), $\mathbb{E}(n - {}_p\overline{X}_n^{(\infty)}) < \infty$. Therefore we can apply the *dominated* *convergence theorem* to conclude that $\mathbb{E}(n - {}_p\overline{X}_n^{(i)}) \overset{i}{\nearrow} \mathbb{E}(n - {}_p\overline{X}_n^{(\infty)})$ and thence $\mathbb{E}({}_p\overline{X}_n^{(i)}) \overset{i}{\searrow} \mathbb{E}({}_p\overline{X}_n^{(\infty)})$. □

Now we can prove the main result of this paper:

Theorem 3.8 (The Convergence Theorem) $p_c^{(i)} \overset{i}{\nearrow} p_c$.

Proof: Suppose $\lim_{i\to\infty} p_c^{(i)} < p_c$ and pick $p \in (\lim_{i\to\infty} p_c^{(i)}, p_c)$. Then Lemma 3.2 and Corollary 3.5 yield $(\forall i \in \mathbb{N})\ M^{(i)}(p) > M^{(i)}(p_c^{(i)}) \overset{\text{def}}{=} 0$. Moreover Lemma 3.6 implies $(\forall i, n \in \mathbb{N})\ \mathbb{E}({}_p\overline{X}_n^{(i)}) \geq n \cdot M^{(i)}(p) \geq 0$. Thus Lemma 3.7 yields

$$(\forall n \in \mathbb{N})\ \mathbb{E}\big({}_p\overline{X}_n^{(\infty)}\big) = \lim_{i\to\infty} \mathbb{E}\big({}_p\overline{X}_n^{(i)}\big) \geq 0 \tag{17}$$

On the other side,

$$\begin{aligned} p < p_c &\Longrightarrow \frac{{}_p\overline{X}_n^{(\infty)}}{n} \xrightarrow{(n)} -\infty \quad a.s. \text{ (Lemma 3.1 and commentary on Lemma 3.2)} \\ &\Longrightarrow \mathbb{E}\left(\frac{{}_p\overline{X}_n^{\infty}}{n}\right) \xrightarrow{(n)} -\infty \quad \text{(Fatou's Lemma applied to } \left(\left(1 - \frac{{}_p\overline{X}_n^{\infty}}{n}\right)_{n\geq 1}\right)\text{)} \\ &\Longrightarrow \exists \bar{n} \in \mathbb{N} : \mathbb{E}\big({}_p\overline{X}_{\bar{n}}^{\infty}\big) < 0 \end{aligned}$$

which contradicts (17). Hence, $\lim_{i\to\infty} p_c^{(i)} \geq p_c$ and the theorem follows from Corollary 3.5 above. □

4. NUMERICAL CALCULATIONS

4.1. Algebraical Determination of the Critical Probabilities

Theorem 3.8 is of theoretical interest by itself. However Corollary 3.5, a weaker result, is enough to show that each $p_c^{(i)}$ $(i \in \mathbb{N})$ is an improved lower bound with regard to its predecessors for the critical probability of *oriented percolation*.

In the sequel, we present the algorithm which determines the *critical probabilities* $p_c^{(i)}$ explicitly (in the case of finite i), although it was already implicit in Sections 2.3.3 and 2.4.

According to Definitions 2.2, 2.3, 2.4, 2.5 and 2.6, each *critical probability* $p_c^{(i)}$, $i \in \mathbb{N}$ may be determined through the following steps:[14]

[14] We refer the reader to the remarks after Definitions 2.3 and 2.5 for details concerning the rational character of the functions below.

(i) **Determination of the Transition Probabilities** $P^{(i)}_{(l,m,k)}(q)$
elementary combinatorics show that the probabilities $P^{(i)}_{(l,m,k)}(q),\ 0 \leq l, m \leq 2^i - 1,\ k \in \mathbb{N}$ may be expressed as polynomial functions of q:

$$P^{(i)}_{(l,m,k)}(q) : \text{polynomial of } q \tag{18}$$

(ii) **Determination of the Transition Matrix** $(q^{(i)}_{lm})_{0 \leq l,m \leq 2^i - 1}$
the transition probabilities defined in (9) may be expressed as

$$q^{(i)}_{lm}(q) = \sum_{k=0}^{\infty} P^{(i)}_{(l,m,k)}(q) \tag{19}$$

Since the numerical sequence $(P^{(i)}_{(l,m,k)}(q))_{k \in \mathbb{N}}$ is a geometric progression (of ratio q^2) except for the first $(i+2)$ terms, the transition probabillities $q^{(i)}_{lm}(q)$ are rational functions of q. As a matter of fact, these probabilities turn out to be polynomials of q:

$$q^{(i)}_{lm}(q) : \text{polynomial of } q$$

(iii) **Determination of the Stationary Measure** $(\pi^{(i)}_l)_{0 \leq l \leq 2^i - 1}$
the combinatorial calculations leading to (18) show that the transition probabilities $q^{(i)}_{lm}(q),\ 0 \leq l, m \leq 2^i - 1$ are strictly positive, provided that $q \in (0, 1)$. Thus the transition matrix $(q^{(i)}_{lm})_{0 \leq l,m \leq 2^i - 1}$ is irreducible and aperiodic and its stationary probability measure $\pi^{(i)}$ in $\mathcal{S}^{(i)}$ can be uniquely determined from $(q^{(i)}_{lm})$ by algebraical means. Therefore

$$\pi^{(i)}_l(q) \text{ is a strictly positive rational function of } q \in (0, 1),\ 0 \leq l \leq 2^i - 1$$

(iv) **Determination of** $M^{(i)}_l$**, the Mean Jump on State** σ_l
Definition 2.4 and the fact that the numerical sequence $(P^{(i)}_{(l,m,k)}(q))_{k \in \mathbb{N}}$ is eventually a geometric progression yield that

$$M^{(i)}_l(q) \text{ is a rational function of } q,\ 0 \leq l \leq 2^i - 1$$

(v) **Determination of** $M^{(i)}$**, the Mean Jump/Drift of** ${}_pX^{(i)}_{\bullet}$
Definition 2.5 and steps (iii) and (iv) above ensure that

$$M^{(i)}(q) \text{ is a rational function of } q$$

(vi) **Determination of the Critical Probabilities** $(p^{(i)}_c)_{i \in \mathbb{N}}$
according to Corollary 3.3 and Definition 2.6, $p^{(i)}_c$ is the only real root of $M^{(i)}(p)$ in $(0, 1)$, hence it must be the only real root of the polynomial in the numerator of $M^{(i)}(p)$. Thus, once $M^{(i)}(p)$ is determined algebraically, $p^{(i)}_c$ may be determined numerically.

In the sequel, we employ the algorithm described above to determine the first critical probabilities:

4.1.1. The Critical Probability of Zeroth Order , $p_c^{(0)}$:

In this case $\left|\mathcal{S}^{(i)}\right| = 1$, so ${}_pY^{(i)}_{\bullet}$ is trivial:

$$\begin{aligned}
M^{(0)}(q) = M_0^{(0)}(q) &= \sum_{k=0}^{\infty}(1-2k).P_{0,0,k}^{(0)}(q) = 1.p - 1.q(1-q^2) \\
&\quad - 3.q^3(1-q^2) - 5.q^5(1-q^2) - \dots \\
&= (1-q) - q(1-q^2).\left[1 + 3q^2 + 5q^4 + 7q^6 + \dots\right] \\
&= (1-q) - q(1-q^2)\frac{1+q^2}{(1-q^2)^2} \\
&= (1-q) - \frac{q+q^3}{1-q^2} = \frac{1-2q-q^2}{1-q^2}
\end{aligned}$$

Hence

$$M^{(0)}(q) = 0 \Leftrightarrow 1 - 2q - q^2 = 0 \Rightarrow p_c^{(0)} = 2 - \sqrt{2} = 0.58579\dots$$

4.1.2. The Critical Probability of First Order, $p_c^{(1)}$:

$$\begin{aligned}
(q_{lm}^{(1)})(q) &= \begin{bmatrix} \dfrac{q - q^3 + q^4}{q^2} & \dfrac{1 - q + q^3 - q^4}{1 - q^2} \end{bmatrix}, \\
\pi_l^{(1)}(q) &= \frac{\left(q^2, 1 - q + q^3 - q^4\right)}{1 - q + q^2 + q^3 - q^4} \\
M_0^{(1)}(q) &= \frac{1 - 2q - 3q^2 + 2q^4}{1 - q^2}, \quad M_1^{(1)}(q) = \frac{1 - 2q - q^2}{1 - q^2}, \\
M^{(1)}(q) &= \frac{1 - 3q + 2q^2 - 6q^4 + q^5 + 3q^6}{1 - q + 2q^3 - 2q^4 - q^5 + q^6}
\end{aligned}$$

Hence

$$M^{(1)}(q) = 0 \Rightarrow p_c^{(1)} = 0.604233\dots$$

4.1.3. The Critical Probability of Second Order, $p_c^{(2)}$:

$$\left(q_{lm}^{(2)}(q)\right) = \begin{bmatrix} q - q^3 + q^6 & q - 2q^2 + q^3 + q^4 - q^6 & 1 - 2q + q^2 + q^4 - q^6 & q^2 - 2q^4 + q^6 \\ q^2 - q^3 + q^4 - q^5 + q^6 & q - q^2 + q^5 - q^6 & q - 2q^2 + 2q^3 - q^4 + q^5 - q^6 & 1 - 2q + 2q^2 - q^3 - q^5 + q^6 \\ 2q^3 - q^4 - q^5 + q^6 - q^7 + q^8 & 2q^2 - 3q^3 + q^4 + q^7 - q^8 & q - 2q^3 + 2q^5 - q^6 + q^7 - q^8 & 1 - q - 2q^2 + 3q^3 - q^5 - q^7 - q^8 \\ q^4 & q^2 - q^4 & q^2 - q^4 & 1 - 2q^2 + q^4 \end{bmatrix}$$

$$\pi_0^{(2)}(q) = \frac{-2q^4 + 2q^5 - 4q^6 + 9q^7 - 14q^8 + 15q^9 - 9q^{10} + 2q^{11}}{-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 - 20q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13}}$$

$$\pi_1^{(2)}(q) = \frac{-q^2 + 2q^3 - 2q^4 + q^5 - 4q^6 + 9q^7 - 5q^8 - 6q^9 + 12q^{10} - 8q^{11} + 2q^{12}}{-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 - 20q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13}}$$

$$\pi_2^{(2)}(q) = \frac{-q^2 + q^3 + 2q^5 - 9q^6 + 15q^7 - 14q^8 + 8q^9 - 2q^{10}}{-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 - 20q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13}}$$

$$\pi_3^{(2)}(q) = \frac{-1 + 3q - 4q^2 + 5q^3 - 13q^4 + 25q^5 - 27q^6 + 13q^7 + 13q^8 - 34q^9 + 37q^{10} - 26q^{11} + 11q^{12} - 2q^{13}}{-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 - 20q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13}}$$

$$M_l^{(2)} = \frac{1}{1-q^2}\left(1 - 2q - 5q^2 + 4q^4, 1 - 2q - 3q^2 + 2q^4, 1 - 2q - q^2 - 2q^4 + 2q^6, 1 - 2q - q^2\right)$$

$$M^{(2)}(q) = \frac{-1 + 5q - 11q^2 + 16q^3 - 25q^4 + 52q^5 - 75q^6 + 96q^7 - 58q^8 - 69q^9 + 152q^{10} - 111q^{11} - 5q^{12} + 74q^{13} - 49q^{14} + 10q^{15}}{\left(1 - q^2\right)\left(-1 + 3q - 6q^2 + 8q^3 - 17q^4 + 30q^5 - 44q^6 + 46q^7 - 28q^8 - 17q^9 + 38q^{10} - 32q^{11} + 13q^{12} - 2q^{13}\right)}$$

Hence

$$M^{(2)}(q) = 0 \Rightarrow p_c^{(2)} = 0.614187\ldots$$

4.2. Numerical Determination of the Critical Probabilities:

The combinatorial calculations leading to the transition matrix $(q^{(i)})_{lm}$ and to the mean-drift vector $M_l^{(i)}$ described in Section 4.1 above, yet fastidious for humans, are taylor-made for computers: while it took us a whole afternoom for calculating $(q^{(2)})_{lm}$ algebraically (16 polynomial entries), a FORTRAN 77 program running at 600 MHz, calculated $(q^{(9)})_{lm}$ (4^9 polinomial entries) in less than one minute.

The biggest problem in writing a computer program version for the algorithm described in Section 4.1 arises precisely in step (iii), viz. the determination of the stationary measure $(\pi_l^{(i)})(q)$ in algebraical terms starting from the transition matrix $(q_{lm}^{(i)})(q)$. True, it could be done straightforwardly, acting on the polynomials as if they were numbers. However, if we don't take into account the fortuitous simplifications that may occur, the degree of the resulting polynomials soon get unmanageable and finding out these simplifications is far from obvious.

Therefore we have adopted a different approach:

(i) Initially the program determined the transition matrix $(q_{lm}^{(i)})$ algebraically in terms of $2^i \times 2^i$ polynomial entries of the non-infection probability q. Moreover, it also determined the mean jump vector $(M_l^{(i)})$ algebraically in terms of 2^i rational functions of q. In other words, we *taught* the program how to perform steps (i), (ii) and (iv) in Section 4.1 above, exactly as we did for small values of i.

(ii) Next, it generated numerical transition matrices and numerical mean jump vectors associated to a decreasing sequence of numerical values for q.

(iii) From each numerical transition matrix, a numerical stationary measure was obtained solving a system of linear equations.[15]

(iv) The numerical *mean-drift* was then obtained performing the inner-product of Definition 2.5.

(v) According to Definition 2.6 $q_c^{(i)} \stackrel{\text{def}}{=} 1 - p_c^{(i)}$ lies between the last value of q for which $M^{(i)}(q)$ is negative and the first value of q for which $M^{(i)}(q)$ is positive.

This approach has the drawback of introducing numerical rounding errors (basically in step (iii) above), a difficulty we had not experienced hitherto. Therefore in order to produce reliable numerical data, rigorous upper bounds for rounding errors should be provided. We did so following the technique of *foward analysis* described in [6]. Basically it consists in determining upper bounds for rounding errors at each single arithmetic operation performed in the flow of numerical calculations leading to the final result. This rather crude approach has the advantage of providing reliable upper bounds for rounding errors, regardless of the particular features the linear system may present.

As a simple example of forward analysis *in action*, suppose we knew that the actual numerical value of x were given by the expression: $x = (1.2 \pm 0.1) \cdot$

[15] At this stage, *partial pivoting*, as described in [6], was of fundamental importance in keeping numerical rouding errors under controll.

Table I. Numerical data related to Subsection 4.2.

$i=1$: p	$M_{Num}^{(1)}(p)$	$\Delta M^{(1)}(p)$	$i=2$: p	$M_{Num}^{(2)}(p)$	$\Delta M^{(2)}(p)$	$i=3$: p	$M_{Num}^{(3)}(p)$	$\Delta M^{(3)}(p)$
0.604231	−9.15E-6	6.2E-16	0.614185	−1.01E-5	6.1E-15	0.620203	−1.36E-5	2.5E-14
0.604232	−5.27E-6	6.2E-16	0.614186	−5.85E-6	6.2E-15	0.620204	−8.96E-6	2.5E-14
0.604233	−1.40E-6	6.2E-16	0.614187	−1.57E-6	6.1E-15	0.620205	−4.32E-6	2.6E-14
0.604234	+2.47E-6	6.2E-16	0.614188	+2.71E-6	6.2E-15	0.620206	+3.15E-7	2.6E-14

$i=4$: p	$M_{Num}^{(4)}(p)$	$\Delta M^{(4)}(p)$	$i=5$: p	$M_{Num}^{(5)}(p)$	$\Delta M^{(5)}(p)$	$i=6$: p	$M_{Num}^{(6)}(p)$	$\Delta M^{(6)}(p)$
			0.627064	−1.31E-5	1.9E-13	0.629201	−1.55E-5	3.8E-13
0.624210	−9.52E-6	8.0E-14	0.627065	−7.85E-6	1.9E-13	0.629202	−9.98E-6	3.8E-13
0.624211	−4.57E-6	8.0E-14	0.627066	−2.63E-6	1.9E-13	0.629203	−4.51E-6	3.9E-13
0.624212	+3.71E-7	8.0E-14	0.627067	+2.60E-6	1.9E-13	0.629204	+9.66E-7	3.8E-13

$i=7$: p	$M_{Num}^{(7)}(p)$	$\Delta M^{(7)}(p)$	$i=8$: p	$M_{Num}^{(8)}(p)$	$\Delta M^{(8)}(p)$	$i=9$: p	$M_{Num}^{(9)}(p)$	$\Delta M^{(9)}(p)$
						0.633260	−1.29E-4	2.0E-12
0.630863	−1.01E-5	7.2E-13	0.632192	−8.53E-6	1.3E-12	0.633270	−6.79E-5	2.0E-12
0.630864	−4.41E-6	7.2E-13	0.632193	−2.60E-6	1.2E-12	0.633280	−6.68E-6	2.0E-12
0.630865	+1.30E-6	7.2E-13	0.632194	+3.32E-6	1.3E-12	0.633290	+5.45E-5	2.0E−12

i	0	1	2	3	4	5	6	7	8	9
$p_c^{(i)}$	0.585787	0.604233	0.614187	0.620205	0.624211	0.627066	0.629203	0.630864	0.632193	0.63328

Table II. Critical probabilities obtained by means of simulation.

i	$p_c^{(i)}$
5	0.627
6	0.629
9	0.6332
20	0.638
40	0.641
100	0.643
200	0.6438
1000	0.64451

(1.2 ± 0.1). Working with a hypothetical two-digit computer, the basic procedure would be as follows:

- calculate the least (x_l), the greatest (x_g) and the average (x_a) numerical possibilities for x:

$$x_l = 1.1 \cdot 1.1 = 1.21 \to 1.2 \qquad x_a = 1.2 \cdot 1.2 = 1.44 \to 1.4$$

$$x_g = 1.3 \cdot 1.3 = 1.69 \to 1.7$$

- determine the upper bound for the correponding rounding error Δx by the formula:

$$\Delta x = \max\{(x_a - x_l), (x_g - x_a)\} = \max\{0.3\ , 0.2\} = 0.3$$

- write the output as $(x_a \pm \Delta x) = (1.4 \pm 0.3)$

At the end, we could state rigorously[16] that $x \in [1.1, 1.7]$.

The numerical data produced following steps i-v above are presented in Table 1:[17]

Therefore we can state that $\boxed{p_c \geq p_c^{(9)} = 0.63328}$

5. SIMULATIONS

Definition 2.1 can be read as a computer algorithm for simulating the right edge process ${}_p\overline{X}_n^{(i)}$. According to Lemma 3.1 the right edge mean speed at time n ${}_p\overline{X}_n^{(i)}/n$ converges to $M^{(i)}(p)$ almost surely, as n tends to infinity. Therefore it is

[16] This turns out to be always the case, provided that the roundings corresponding to x_g are performed upwards, whereas the roundings corresponding to x_l are performed downwards.

[17] the calculations were carried out on double precision. $|M^{(i)}(p) - M^{(i)}_{Num}(p)| \leq \Delta M^{(i)}(p)$; where $M^{(i)}_{Num}(p)$ denotes the *asymptotic mean drift* of i^{th} order determined numerically.

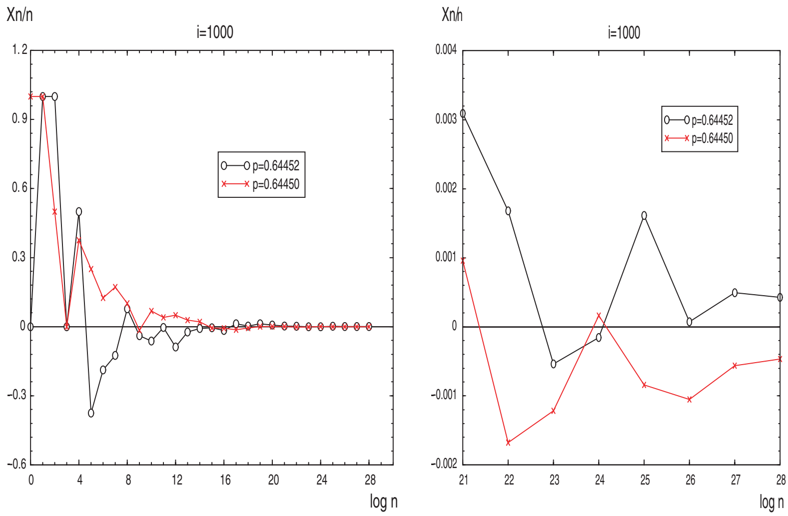

Fig. 5. Independent realizations of the processes ${}_{.64450}\overline{X}_n^{(1000)}/n$ ${}_{.64452}\overline{X}_n^{(1000)}/n$ and enlarged detail.

possible to determine $p_c^{(i)}$ observing the height of the plateau that takes shape as n increases, according with the rule:

- *height of plateau* $< 0 \Rightarrow p < p_c^{(i)}$,
- *height of plateau* $> 0 \Rightarrow p > p_c^{(i)}$.

Figure 5 ilustrates the use of this simulation technique in the case of $i = 1000$.

It is worth observing that the *plateau* pattern of Fig. 5 occurs even for large values of i, where $p_c^{(i)} \approx p_c$. Therefore it is possible to generate sharp lower estimates for p_c running the algorithm until the plateau pattern takes shape.

Table 2 summarizes our simulation results.

ACKNOWLEDGMENTS

The authors are thankful to the referees of the *Journal of Statistical Physics* for their useful suggestions and to M. Menshikov, who suggested the use of *Markov chains in a strip* to *track* the edge of oriented percolation.

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